Max-Planck-Institut für Plasmaphysik

A three-dimensional reduced MHD model consistent with full MHD

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Acknowledgements to: Omar Maj, Florian Hindenlang, Per Helander, Boniface Nkonga, Guido Huijsmans, Erika Strumberger, Alessandro Biancalani and Rohan Ramasamy

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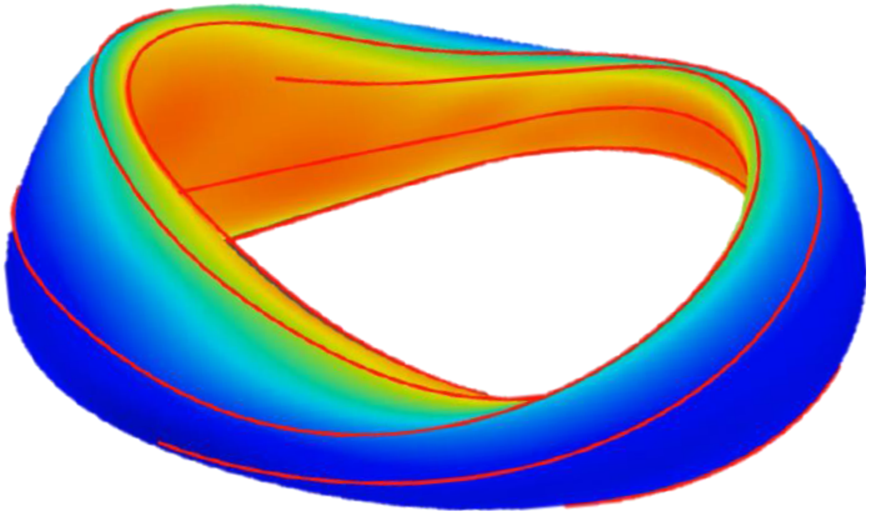

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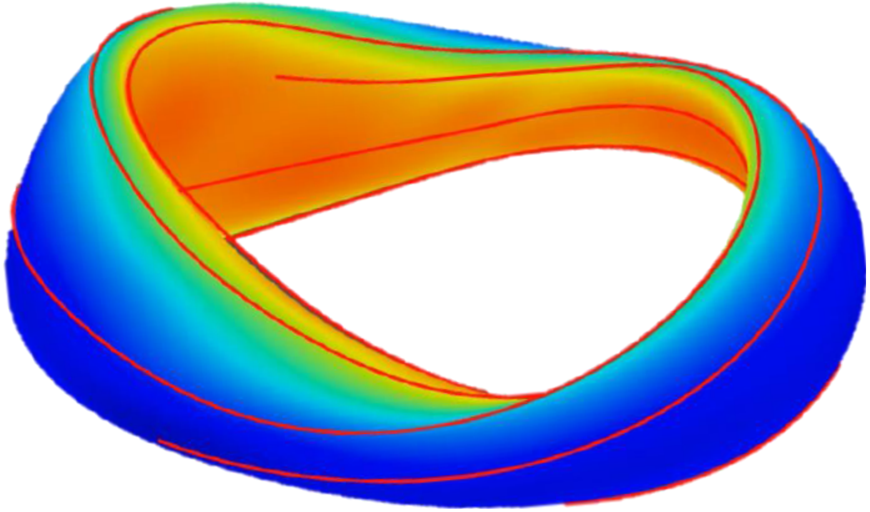

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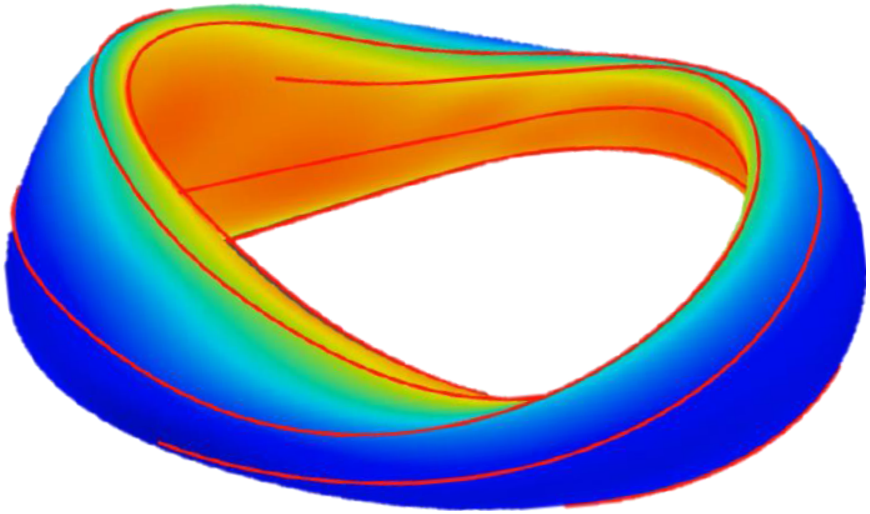

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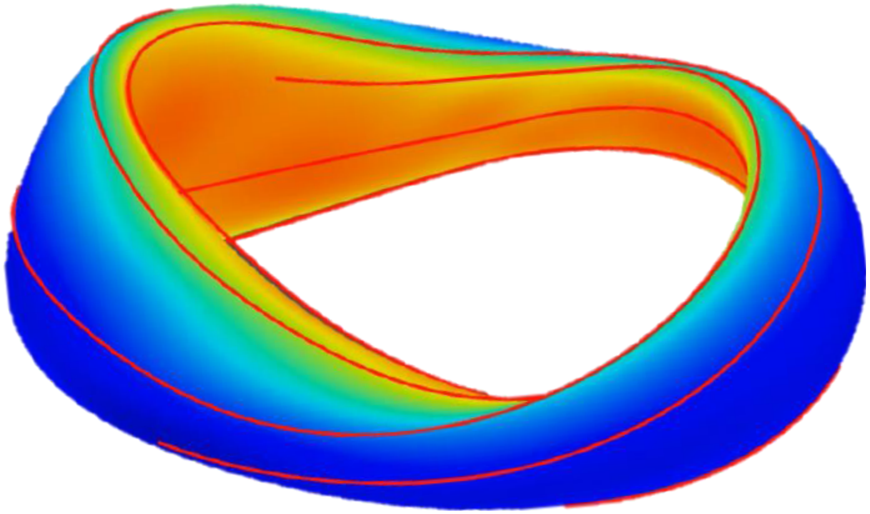

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- The presence of a significant plasma current raises the question of whether quasiaxisymmetric stellarators are prone to disruptions
- For the present work, we derive reduced MHD models suitable for stellarator geometries. This aims at future computational studies addressing the question of disruptions.

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- This model is mostly a generalization of the tokamak-specific model by Breslau et al (Steven Jardin's group, PPPL) and a similar model by Izzo et al

Starting point equations

- We take the viscoresistive MHD equations with anisotropic heat flow and mass diffusion, as shown above, as the base equations and begin our derivation from them

Starting point equations

Continuity: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = P,$$

Navier-Stokes: $$\frac{\partial}{\partial t}(\rho \vec{v}) + \nabla \cdot (\rho \vec{v}\vec{v}) = \vec{j} \times \vec{B} - \nabla p + \rho \nu \Delta \vec{v},$$

Energy:

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0}\right) + \nabla \cdot \left[\left(\frac{\rho v^2}{2} + \frac{\gamma p}{\gamma - 1}\right)\vec{v} + \frac{p}{\gamma - 1}\frac{D_\perp}{\rho}\nabla_\perp \rho + \frac{\vec{E} \times \vec{B}}{\mu_0} - \kappa_\perp \nabla_\perp T - \kappa_\parallel \nabla_\parallel T\right] = S_e - \frac{v^2}{2}P,$$

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$$\nabla_\parallel = \frac{\vec{B}}{B^2}\vec{B} \cdot \nabla \qquad \nabla_\perp = \nabla - \nabla_\parallel \qquad \partial^\parallel = \frac{\nabla \chi}{B_v} \cdot \nabla \qquad \nabla^\perp = \nabla - \frac{\nabla \chi}{B_v}\partial^\parallel$$

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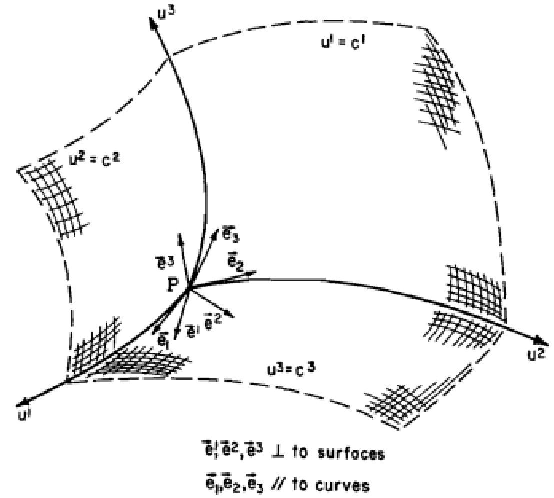
- The curvilinear non-orthogonal coordinate system *(α, β, χ)* is a Clebsch-type system aligned to ***B***

$$\vec{e}^{\,\alpha} \equiv \nabla\alpha, \qquad \vec{e}^{\,\beta} \equiv \nabla\beta, \qquad \vec{e}^{\,\chi} \equiv \nabla\chi,$$

$$\vec{e}_\alpha \equiv J\nabla\beta \times \nabla\chi, \quad \vec{e}_\beta \equiv J\nabla\chi \times \nabla\alpha, \quad \vec{e}_\chi \equiv J\nabla\alpha \times \nabla\beta.$$



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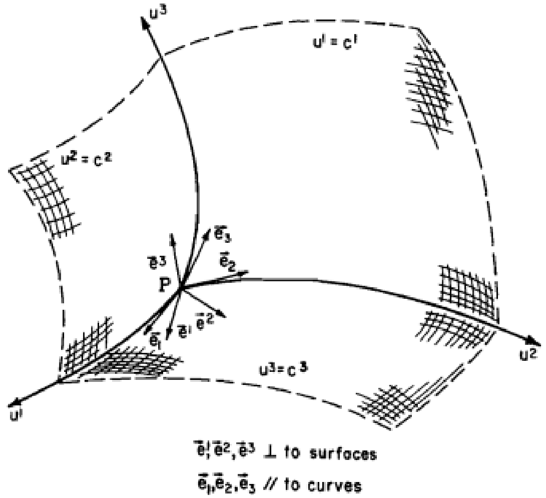

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- We will denote the vacuum field as $\boldsymbol{B}_v = \nabla\chi = \nabla\psi_v \times \nabla\beta_v$ and the total field as $\boldsymbol{B} = \boldsymbol{B}_v + \boldsymbol{B}_{ind} = \nabla\alpha \times \nabla\beta$

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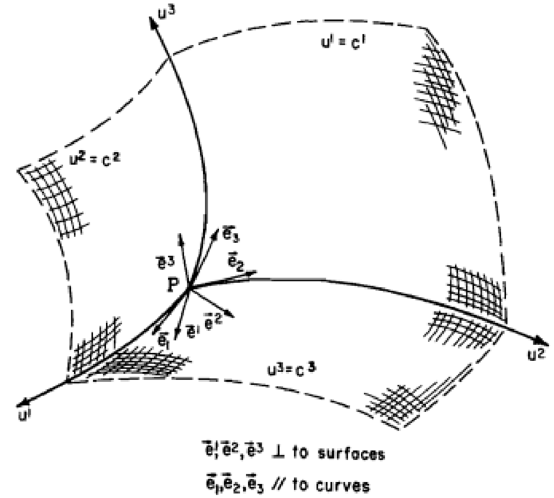

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- There is another coordinate system (ψ_v, β_v, χ) which is aligned to $\boldsymbol{B}_v$; we use ***b*** for this system's basis vectors

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  - Terms contain Alfven waves, slow magnetosonic waves and fast magnetosonic waves

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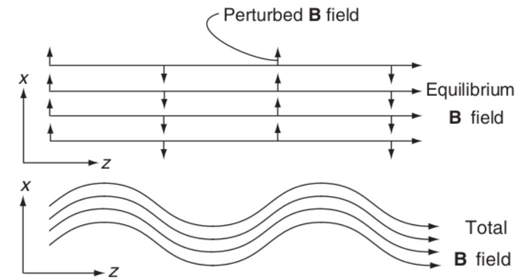

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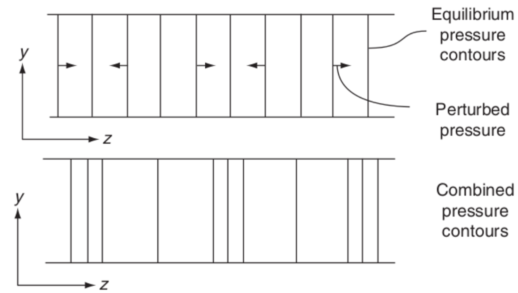

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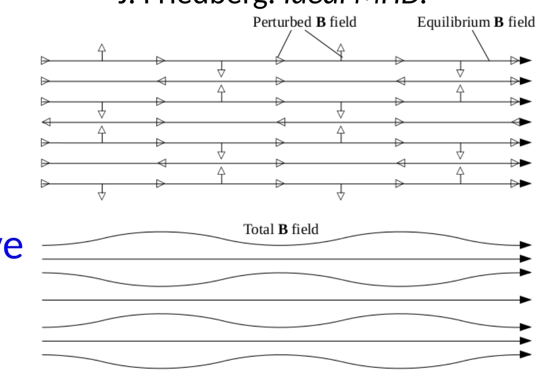

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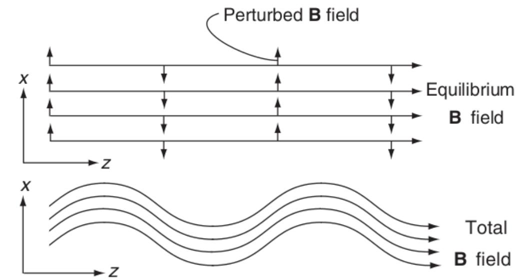
  - First and last terms both capture pressure- and current-driven instabilities

Alfven wave



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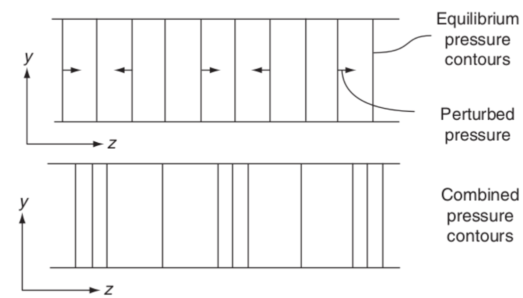

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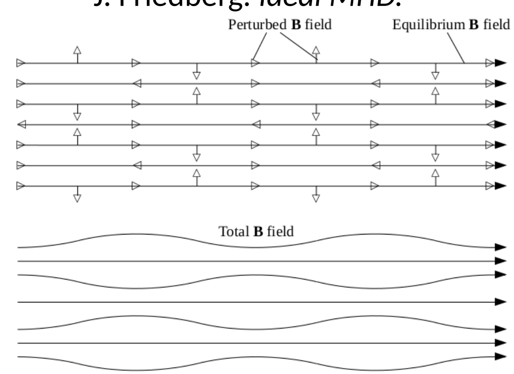

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- Commutator between projection operator and $\nu\Delta$ neglected
- Resulting equations are still full MHD

Resulting equations (full MHD)

Continuity:

$$\frac{\partial\rho}{\partial t} = -B_v\left[\frac{\rho}{B_v^2},\Phi\right] - B_v\partial^{\|}(\rho v_{\|}) - B_v[\rho v_{\|},\Psi] - F_v[\rho v_{\|},\Omega]_{\psi_v} - (\rho,\zeta) - \rho\Delta^{\perp}\zeta + P,$$

Energy:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}+\frac{p}{\gamma-1}+\frac{B^2}{2\mu_0}\right) &= -B_v\left[\frac{1}{B_v^2}\left(\frac{\rho v^2}{2}+\frac{\gamma p}{\gamma-1}+\frac{B^2}{\mu_0}\right),\Phi\right] - B_v\partial^{\|}\left[v_{\|}\left(\frac{\rho v^2}{2}+\frac{\gamma p}{\gamma-1}+\frac{B^2}{\mu_0}\right)-\frac{\vec{v}\cdot\vec{B}}{\mu_0}\right] \\
&- B_v\left[v_{\|}\left(\frac{\rho v^2}{2}+\frac{\gamma p}{\gamma-1}+\frac{B^2}{\mu_0}\right)-\frac{\vec{v}\cdot\vec{B}}{\mu_0},\Psi\right] - F_v\left[v_{\|}\left(\frac{\rho v^2}{2}+\frac{\gamma p}{\gamma-1}+\frac{B^2}{\mu_0}\right)-\frac{\vec{v}\cdot\vec{B}}{\mu_0},\Omega\right]_{\psi_v} - \left(\frac{\rho v^2}{2}+\frac{\gamma p}{\gamma-1}+\frac{B^2}{\mu_0},\zeta\right) \\
&- \left(\frac{\rho v^2}{2}+\frac{\gamma p}{\gamma-1}+\frac{B^2}{\mu_0}\right)\Delta^{\perp}\zeta - \frac{1}{\mu_0}\nabla\cdot(\eta\vec{j}\times\vec{B}) + \nabla\cdot\left[\frac{p}{\gamma-1}\frac{D_{\perp}}{\rho}\left(\nabla\rho - \frac{\vec{B}}{B^2}(B_v\partial^{\|}\rho + B_v[\rho,\Psi] + F_v[\rho,\Omega]_{\psi_v})\right)\right. \\
&\left. + \frac{\kappa_{\perp}}{R}\nabla\left(\frac{p}{\rho}\right) + \frac{(\kappa_{\|}-\kappa_{\perp})\vec{B}}{RB^2}\left(B_v\partial^{\|}\left(\frac{p}{\rho}\right) + B_v\left[\frac{p}{\rho},\Psi\right] + F_v\left[\frac{p}{\rho},\Omega\right]_{\psi_v}\right)\right] + S_e - \frac{v^2}{2}P,
\end{aligned}$$

Magnetic potential:

$$\left[\psi_v,\frac{\partial\Psi}{\partial t}\right] = \left[\frac{[\Psi,\Phi]-\partial^{\|}\Phi}{B_v},\psi_v\right] - \frac{F_v}{B_v}\left[\Omega,\frac{[\psi_v,\Phi]}{B_v}\right]_{\psi_v} + \partial^{\|}(F_v\partial_{\psi_v}\zeta) + [(\zeta,\Psi),\psi_v] - \frac{F_v}{B_v}[\Omega,(\zeta,\psi_v)]_{\psi_v} + \frac{1}{B_v}\nabla\cdot(\eta\nabla\psi_v\times\vec{j}),$$

$$\left[\frac{\partial\Omega}{\partial t},\psi_v\right] = -\left[\Omega,\frac{[\psi_v,\Phi]}{B_v}\right] + \left[\psi_v,\frac{[\Omega,\Phi]}{B_v}\right] - 2(B_v,\zeta) - B_v\Delta^{\perp}\zeta - [\Omega,(\zeta,\psi_v)] + [\psi_v,(\zeta,\Omega)] + \frac{1}{B_v}\nabla\cdot(\eta\nabla\chi\times\vec{j})$$

Notation:

$$\partial_{\psi_v} = \frac{\nabla\psi_v}{F_v}\cdot\nabla \qquad \partial^{\|} = \frac{\nabla\chi}{B_v}\cdot\nabla \qquad \nabla^{\perp} = \nabla - \frac{\nabla\chi}{B_v}\partial^{\|} \qquad \Delta^{\perp} = \nabla\cdot\nabla^{\perp}$$

$$[a,b] = \frac{\nabla\chi}{B_v}\cdot(\nabla a\times\nabla b) \qquad (a,b) = \nabla^{\perp}a\cdot\nabla^{\perp}b \qquad [a,b]_{\psi_v} = \frac{\nabla\psi_v}{F_v}\cdot(\nabla a\times\nabla b) \qquad (a,b)_{\psi_v} = \nabla a\cdot\nabla b - \partial_{\psi_v}a\,\partial_{\psi_v}b$$

Resulting equations (full MHD, continued)

Φ equation:

$$
\begin{aligned}
&\Delta^\perp \frac{\partial\Phi}{\partial t} + \left(v_\| B_v^2, \frac{\partial\Psi}{\partial t}\right) + v_\| B_v^2 \Delta^\perp \frac{\partial\Psi}{\partial t} - F_v \partial_{\psi_v}\left(v_\| B_v \partial^\| \frac{\partial\Omega}{\partial t}\right) - v_\| B_v \partial^\| \frac{\partial\Omega}{\partial t}\Delta\psi_v - \left(\frac{v_\| B_v^2}{B_v+[\Omega,\psi_v]}\left[\frac{\partial\Omega}{\partial t},\psi_v\right],\Psi\right) \\
&- \frac{v_\| B_v^2}{B_v+[\Omega,\psi_v]}\left[\frac{\partial\Omega}{\partial t},\psi_v\right]\Delta^\perp\Psi + F_v\partial_{\psi_v}\left(\frac{v_\| B_v\partial^\|\Omega}{B_v+[\Omega,\psi_v]}\left[\frac{\partial\Omega}{\partial t},\psi_v\right]\right) + \frac{v_\| B_v\partial^\|\Omega}{B_v+[\Omega,\psi_v]}\left[\frac{\partial\Omega}{\partial t},\psi_v\right]\Delta\psi_v \\
&= \nabla\cdot\Big[\frac{B_v\nabla^\perp\Psi - \partial^\|\Omega\nabla\psi_v}{B_v+[\Omega,\psi_v]}\frac{B_v\partial^\| v^2}{2} + \omega_\chi(\nabla\Phi\times\nabla\chi + B_v^2 v_\|\vec B^\perp + B_v^2\nabla^\perp\zeta) - v_\chi B_v^2\vec\omega^\perp + \frac{B^2 B_v}{\rho(B_v+[\Omega,\psi_v])}\vec j^\perp \\
&- \frac{j_\chi B_v^2}{\rho}\vec B^\perp - (\nabla^\perp\Phi + \nabla\chi\times\nabla\zeta)\frac{P}{\rho}\Big] + B_v\left[\frac{1}{\rho},p\right] + \left(\frac{B_v^2\partial^\| p}{\rho(B_v+[\Omega,\psi_v])},\Psi\right) + \frac{B_v^2\partial^\| p}{\rho(B_v+[\Omega,\psi_v])}\Delta^\perp\Psi \\
&- F_v\partial_{\psi_v}\left(\frac{B_v\partial^\|\Omega\partial^\| p}{\rho(B_v+[\Omega,\psi_v])}\right) - \frac{B_v\partial^\|\Omega\partial^\| p}{\rho(B_v+[\Omega,\psi_v])}\Delta\psi_v + \nu\Delta\Delta^\perp\Phi
\end{aligned}
$$

Shorthand expressions:

$$v^2 = \frac{1}{B_v^2}(\Phi,\Phi) + 2v_\|\vec v\cdot\vec B + \frac{2}{B_v}[\zeta,\Phi] - v_\|^2 B^2 + (\zeta,\zeta),$$

$$\vec v\cdot\vec B = (\Phi,\Psi) - \frac{F_v}{B_v}\partial_{\psi_v}\Phi\partial^\|\Omega + v_\| B^2 + B_v[\zeta,\Psi] + F_v[\zeta,\Omega]_{\psi_v} - \partial^\|\zeta[\Omega,\psi_v],$$

$$B^2 = B_v^2 + 2B_v[\Omega,\psi_v] + B_v^2(\Psi,\Psi) - 2B_v F_v\partial_{\psi_v}\Psi\partial^\|\Omega + F_v^2(\Omega,\Omega)_{\psi_v},$$

$$\vec j = \frac{1}{\mu_0}[-\nabla\chi\Delta\Psi + B_v\partial^\|\nabla\Psi - (\nabla\Psi\cdot\nabla)\nabla\chi + \nabla\Omega\Delta\psi_v - \nabla\psi_v\Delta\Omega + F_v\partial_{\psi_v}\nabla\Omega - (\nabla\Omega\cdot\nabla)\nabla\psi_v].$$

$$P = \nabla\cdot\left[D_\perp\nabla\rho - \frac{D_\perp\vec B}{B^2}(B_v\partial^\|\rho + B_v[\rho,\Psi] + F_v[\rho,\Omega]_{\psi_v})\right] + S_\rho$$

$$\vec\omega = \nabla\times\vec v = -\nabla\chi\nabla\cdot\left(\frac{\nabla\Phi}{B_v^2}\right) + B_v\partial^\|\frac{\nabla\Phi}{B_v^2} - \frac{1}{B_v^2}(\nabla\Phi\cdot\nabla)\nabla\chi + \nabla v_\|\times\vec B + v_\|\vec j + \nabla\chi\times\nabla\frac{\partial^\|\zeta}{B_v}.$$

Resulting equations (full MHD, continued)

$v_{||}$ equation:

$$\frac{\partial v_{||}}{\partial t} + \frac{v_{||}}{B_v + [\Omega, \psi_v]} \left[\frac{\partial \Omega}{\partial t}, \psi_v\right] = -\frac{v_{||}}{\rho} P + \frac{1}{B_v + [\Omega, \psi_v]} \Big[\frac{\vec{\omega} \cdot \nabla^{\perp} \Phi}{B_v} + B_v v_{||} \vec{\omega} \cdot \nabla^{\perp} \Psi - v_{||} \omega^{\psi_v} \partial^{||} \Omega$$

$$- \frac{\vec{\omega} \cdot (\nabla \zeta \times \nabla \chi)}{B_v} - \frac{\partial^{||} v^2}{2} - \frac{B_v \vec{j} \cdot \nabla^{\perp} \Psi + j^{\psi_v} \partial^{||} \Omega - \partial^{||} p}{\rho}\Big] + \nu \Delta v_{||},$$

ζ equation:

$$B_v^2 \Delta^{\perp} \frac{\partial \zeta}{\partial t} + \left(B_v^2, \frac{\partial \zeta}{\partial t}\right) + B_v \left[v_{||} B_v^2, \frac{\partial \Psi}{\partial t}\right] + F_v \left[v_{||} B_v^2, \frac{\partial \Omega}{\partial t}\right]_{\psi_v} - B_v \partial^{||} \left(\frac{v_{||} B_v^2}{B_v + [\Omega, \psi_v]} \left[\frac{\partial \Omega}{\partial t}, \psi_v\right]\right)$$

$$- B_v \left[\frac{v_{||} B_v^2}{B_v + [\Omega, \psi_v]} \left[\frac{\partial \Omega}{\partial t}, \psi_v\right], \Psi\right] - F_v \left[\frac{v_{||} B_v^2}{B_v + [\Omega, \psi_v]} \left[\frac{\partial \Omega}{\partial t}, \psi_v\right], \Omega\right]_{\psi_v} = \nabla \cdot \Big[B_v^2 v_\chi \nabla \chi \times \vec{\omega} - \frac{B_v^2}{2} \nabla v^2$$

$$- B_v^2 \omega_\chi (\nabla^{\perp} \Phi + v_{||} B_v^2 \nabla^{\perp} \Psi - v_{||} B_v \partial^{||} \Omega \nabla \psi_v + \nabla \chi \times \nabla \zeta) - (\nabla \Phi \times \nabla \chi + B_v^2 \nabla^{\perp} \zeta) \frac{P}{\rho} - \frac{B_v B^2}{\rho(B_v + [\Omega, \psi_v])} \nabla \chi \times \vec{j}$$

$$+ \frac{B_v^3 j_\chi}{\rho} (B_v \nabla^{\perp} \Psi - \partial^{||} \Omega \nabla \psi_v) + \frac{B_v^2 (\nabla \chi + \nabla \Psi \times \nabla \chi + \nabla \Omega \times \nabla \psi_v) \partial^{||} v^2}{2(B_v + [\Omega, \psi_v])}\Big] - \frac{B_v^2}{\rho} \Delta p - \nabla \left(\frac{B_v^2}{\rho}\right) \cdot \nabla p$$

$$+ B_v \partial^{||} \left(\frac{B_v^2 \partial^{||} p}{\rho(B_v + [\Omega, \psi_v])}\right) + B_v \left[\frac{B_v^2 \partial^{||} p}{\rho(B_v + [\Omega, \psi_v])}, \Psi\right] + F_v \left[\frac{B_v^2 \partial^{||} p}{\rho(B_v + [\Omega, \psi_v])}, \Omega\right]_{\psi_v} + \nu B_v^2 \Delta \Delta^{\perp} \zeta.$$

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- Set $\zeta = \Omega = 0$ to remove fast magnetosonic waves and magnetic field compression
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$$\vec{B} = \nabla \chi + \nabla \Psi \times \nabla \chi,$$

$$\vec{v} = \frac{\nabla \Phi \times \nabla \chi}{B_v^2} + v_{\|} \vec{B},$$

$$\vec{j} = \frac{1}{\mu_0}[-\nabla \chi \Delta \Psi + B_v \partial^{\|} \nabla \Psi - (\nabla \Psi \cdot \nabla) \nabla \chi],$$

$$\vec{\omega} = \nabla \times \vec{v} = -\nabla \chi \nabla \cdot \left(\frac{\nabla \Phi}{B_v^2} \right) + B_v \partial^{\|} \frac{\nabla \Phi}{B_v^2} - \frac{1}{B_v^2}(\nabla \Phi \cdot \nabla) \nabla \chi + \nabla v_{\|} \times \vec{B} + v_{\|} \vec{j}.$$

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- In some cases, one can also set $v_{||} = 0$ and drop the $v_{||}$ equation

Reduced equations

Continuity

$$\frac{\partial \rho}{\partial t} = -B_v \left[\frac{\rho}{B_v^2}, \Phi\right] - B_v \partial^{\|}(\rho v_{\|}) - B_v[\rho v_{\|}, \Psi] + P,$$

Energy:

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0}\right) = -B_v \left[\frac{1}{B_v^2}\left(\frac{\rho v^2}{2} + \frac{\gamma p}{\gamma - 1} + \frac{B^2}{\mu_0}\right), \Phi\right] - B_v \partial^{\|}\left[v_{\|}\left(\frac{\rho v^2}{2} + \frac{\gamma p}{\gamma - 1} + \frac{B^2}{\mu_0}\right) - \frac{\vec{v}\cdot\vec{B}}{\mu_0}\right]$$

$$- B_v \left[v_{\|}\left(\frac{\rho v^2}{2} + \frac{\gamma p}{\gamma - 1} + \frac{B^2}{\mu_0}\right) - \frac{\vec{v}\cdot\vec{B}}{\mu_0}, \Psi\right] + \nabla\cdot\left[\frac{p}{\gamma-1}\frac{D_\perp}{\rho}\left(\nabla\rho - \frac{\vec{B}}{B^2}(B_v\partial^{\|}\rho + B_v[\rho, \Psi])\right)\right.$$

$$\left. + \frac{\kappa_\perp}{R}\nabla\left(\frac{p}{\rho}\right) + \frac{(\kappa_{\|} - \kappa_\perp)\vec{B}}{RB^2}\left(B_v\partial^{\|}\left(\frac{p}{\rho}\right) + B_v\left[\frac{p}{\rho}, \Psi\right]\right)\right] - \frac{1}{\mu_0}\nabla\cdot(\eta\vec{j}\times\vec{B}) + S_e - \frac{v^2}{2}P$$

Magnetic potential:

$$\left[\psi_v, \frac{\partial\Psi}{\partial t}\right] = \left[\frac{[\Psi,\Phi] - \partial^{\|}\Phi}{B_v}, \psi_v\right] + \frac{1}{B_v}\nabla\cdot(\eta\nabla\psi_v\times\vec{j})$$

Φ equation:

$$\Delta^\perp\frac{\partial\Phi}{\partial t} + \left(v_{\|}B_v^2, \frac{\partial\Psi}{\partial t}\right) + v_{\|}B_v^2\Delta^\perp\frac{\partial\Psi}{\partial t} = \nabla\cdot\left[\frac{B_v\partial^{\|}v^2}{2}\nabla^\perp\Psi + \omega_\chi(\nabla\Phi\times\nabla\chi + B_v^2 v_{\|}\nabla\Psi\times\nabla\chi) - v_\chi B_v^2\vec{\omega}^\perp\right.$$

$$\left. + \frac{B^2}{\rho}\vec{j}^\perp - \frac{j_\chi B_v^2}{\rho}\nabla\Psi\times\nabla\chi - \frac{P}{\rho}\nabla^\perp\Phi\right] + B_v\left[\frac{1}{\rho}, p\right] + \left(\frac{B_v\partial^{\|}p}{\rho}, \Psi\right) + \frac{B_v\partial^{\|}p}{\rho}\Delta^\perp\Psi + \nu\Delta\Delta^\perp\Phi$$

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- Mass and energy conservation are exact since those equations are used directly, without neglecting terms
- Components of Faraday's law and Navier-Stokes equation are dropped in reduction
- This leads to non-conservation of momentum and additional non-conservation of flux
- Importantly, reduced MHD approximation is only valid when the dropped terms are small, as discussed on the following slides

Non-conservation of flux

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  – Reduction error is also locally proportional to perpendicular current, which arises due to nonzero parallel derivatives of Ψ and g^{ik}, discussed on the next slide

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  - **Should be a good approximation for most magnetic confinement devices**

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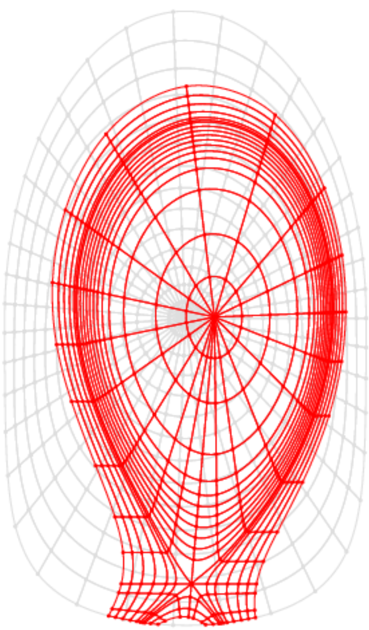
  - In the Φ and $v_{||}$ equations, $\boldsymbol{B}$ is approximated by $\boldsymbol{B}_v$ in the time derivative, Lorentz force term and projection operator
  - Hydrodynamic and ram pressure drop from the $v_{||}$ equation

Summary

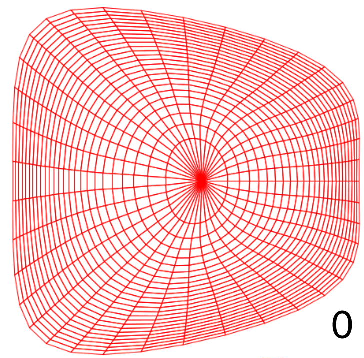
- Derived a hierarchy of MHD models suitable for stellarators: full MHD, reduced MHD with parallel velocity, reduced MHD without parallel velocity
- Also suitable for astrophysical systems with a strong guide field, since toroidal geometry was not assumed in the derivation
- Divergence free field and good conservation properties

Next steps

- Equations are presently being implemented in the JOREK non-linear MHD code [Huysmans and Czarny]
- Tokamak limit should be similar to the present JOREK and will be tested as next step
- Then, extend JOREK grid to 3D and investigate quasi-axisymmetric stellarators



Axisymmetric grids in JOREK.
Flux-aligned grid is in red.



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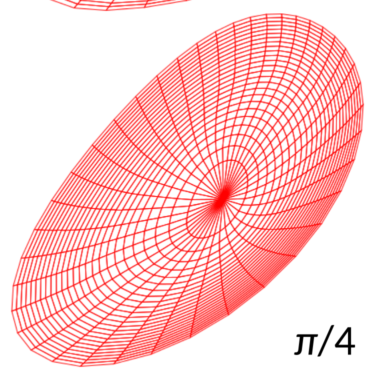
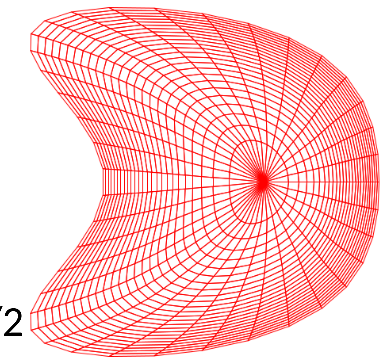
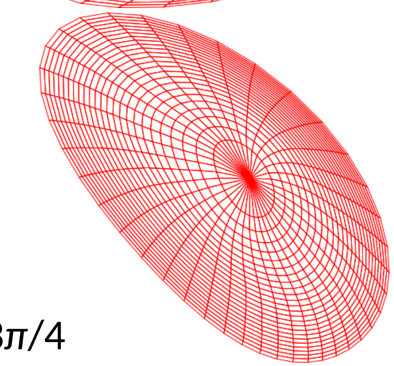


$\pi/4$

Flux-aligned grid for a quasi-axisymmetric stellarator at various toroidal angles. To be implemented.

$\pi/2$





$3\pi/4$

References used

- Henneberg, S., et al, Design of a new quasi-axisymmetric stellarator (presentation, 2018).
- William D D'haeseleer, William NG Hitchon, James D Callen, and J Leon Shohet. *Flux coordinates and magnetic field structure: a guide to a fundamental tool of plasma theory.* Springer Science & Business Media, 2012.
- Friedberg, J., *Ideal MHD*, Cambridge University Press, 2014.
- GTA Huysmans and O Czarny. MHD stability in X-point geometry: simulation of ELMs. *Nuclear Fusion*, 47, 659, 2007.
- J Breslau, N Ferraro, and S Jardin. Some properties of the M3D-C 1 form of the three-dimensional magnetohydrodynamics equations. *Physics of Plasmas*, 16(9):092503, 2009.
- HR Strauss. Reduced MHD in nearly potential magnetic fields. *Journal of Plasma Physics*, 57(1):83–87, 1997.
- Emmanuel Franck, Matthias Hölzl, Alexander Lessig, and Eric Sonnendrücker. Energy conservation and numerical stability for the reduced MHD models of the non-linear JOREK code. *ESAIM: Mathematical Modelling and Numerical Analysis*, 49(5):1331–1365, 2015.
- HR Strauss. Nonlinear, three-dimensional magnetohydrodynamics of noncircular tokamaks. *The Physics of Fluids*, 19(1):134–140, 1976.
- HR Strauss. Dynamics of high β tokamaks. *The Physics of Fluids*, 20(8):1354–1360, 1977.
- HR Strauss. Stellarator equations of motion. *Plasma Physics*, 22(7):733, 1980.
- Scott E Kruger, Chris C Hegna, and James D Callen. Generalized reduced magnetohydrodynamic equations. *Physics of Plasmas*, 5(12):4169–4182, 1998.