

A Gyrokinetic Model for the Plasma Periphery of Tokamak Devices

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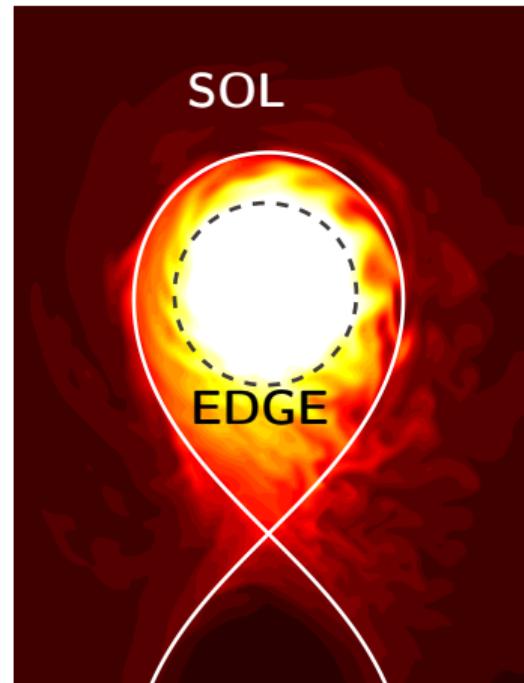
- The Plasma Periphery of Tokamak Devices
- The Gyrokinetic Model for the Plasma Periphery
- Model Application: sITG
- Conclusion

The Plasma Periphery of Tokamak Devices



Edge Region
+
Scrape-Off Layer (SOL)

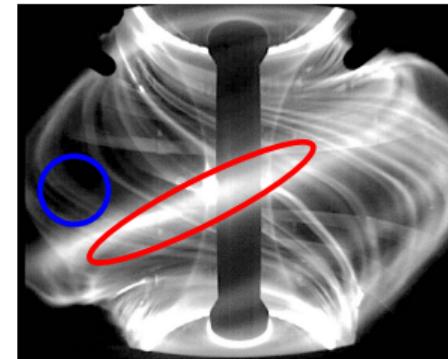
- Particle and Heat Exhaust
- Fueling and Ash Removal
- Confinement (e.g., L-H mode transition)



- Anisotropic ($k_{\parallel}/k_{\perp} \ll 1$) low-frequency ($\omega/\Omega_i \ll 1$) fluctuations

- ▶ $\frac{e\delta\phi}{T_e} \ll 1$ at $k_{\perp}\rho \gtrsim 1$ scales
- ▶ $\frac{e\delta\phi}{T_e} \sim 1$ at $k_{\perp}\rho \ll 1$ scales

- **No separation** between equilibrium and fluctuations
- **Large range of collisionality**, $k_{\parallel}\lambda_{mfp} \ll 1$ & $k_{\parallel}\lambda_{mfp} \gtrsim 1$
- **Challenges** standard δf_a -gyrokinetic theories



$$\frac{\omega}{\Omega_i} \sim \frac{e\delta\phi}{T_e} \sim \frac{\delta f_a}{f_a} \sim \frac{k_{\parallel}}{k_{\perp}} \ll 1, \quad k_{\perp}\rho_i \sim 1$$

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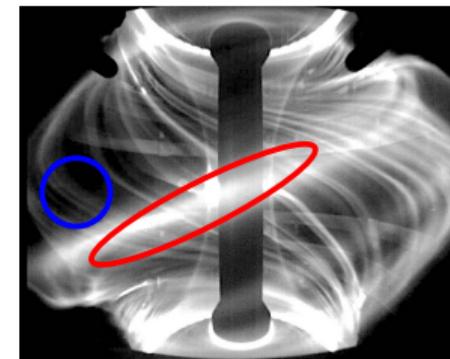
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$$\frac{\omega}{\Omega_i} \sim \frac{e\delta\phi}{T_e} \sim \frac{\delta f_a}{f_a} \sim \frac{k_{\parallel}}{k_{\perp}} \ll 1, \quad k_{\perp}\rho_i \sim 1$$



- Attempt to a **generalized ordering** by Bernstein & Catto (1985), Dimits *et al.* (1992), Hahm *et al.* (1996 and 2009), Qin *et al* (2006 and 2007), Dimits *et al.* (2012), and Madsen (2010 and 2013)

- Cannot handle properly **arbitrary large deviations from equilibrium**, e.g.
 - ▶ Evaluation of Polarization effects (e.g. Hahm *et al.* (2009))

$$\delta n \sim \int d\mathbf{v} \tilde{\phi} \partial_\mu f_{a0}$$

- ▶ Evaluation of FLR effects (e.g. Madsen (2013))

$$\int d\mathbf{v} \mathbf{v}'^I J_0 f_{a0} \simeq \Gamma_0$$

- A **collision operator** fulfilling the plasma periphery conditions:

- ▶ **Full-F Non-Linear operator**

$$C(f_{a0}, \delta f_b), C(\delta f_a, f_{b0}) \sim C(\delta f_a, \delta f_b)$$

- ▶ Multi-species \Rightarrow **Arbitrary m_a/m_b and T_a/T_b ratios**
 - ▶ **Accurate FLR effects**
 - ▶ **Challenging to handle, velocity-space integro-differential operator**

$$C(f_a, f_b) = L_{ab} \nabla_{\mathbf{v}} \cdot \left[\int d\mathbf{v}' \mathbf{U} \cdot \left(\frac{\nabla_{\mathbf{v}} f_a}{m_a} f_b(\mathbf{v}') - \frac{\nabla_{\mathbf{v}'} f_b(\mathbf{v}')}{m_b} f_a(\mathbf{v}) \right) \right]$$

- ▶ **Necessary** at intermediate level of collisionality (e.g. pedestal) (Jorge *et al.* PRL (2018))

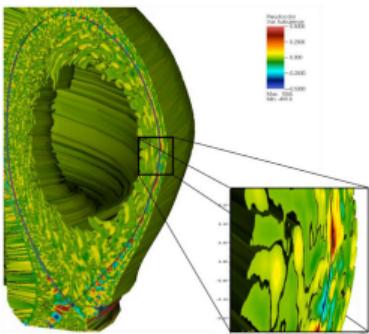
- Self-consistent GK Maxwell's equations:

- ▶ For edge application, **gyrokinetic long wavelength limit (LWL)** (see Xu *et al.* (2007) and Pan *et al.* (2018))

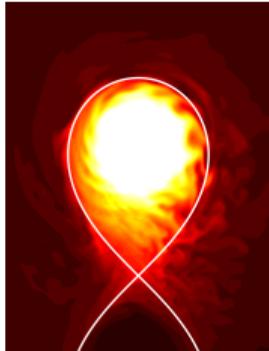
$$\nabla^2 \phi + \sum_a \frac{1}{2} \left(\frac{\rho_a}{\lambda_{Da}} \right)^2 \nabla \cdot (\ln N_a \nabla_{\perp} \phi) = -4\pi \sum_a q_a N_a$$
$$- \sum_a \left(\frac{\rho_a}{\lambda_{Da}} \right)^2 \frac{1}{4q_a N_a} \nabla_{\perp}^2 P_{\perp a}$$

- ▶ Parra & Catto (2008), Lee *et al.* (2009) and Miyato *et al.* (2013) question gyrokinetic LWL approximation
 - ▶ Need for GK theory to treat **self-consistent** polarization and magnetization effects

- **Important recent progress** in GK simulation of periphery (Shi *et al.* (2017), Pan *et al.* (2018), and Neiser *et al.* (2019))
- **Large computational resources** ($3D2V + 1$) (Chang *et al.* (2017))
- Braginskii-like fluid simulations (see, e.g., Paruta *et al.*, (2018))
 - ▶ **Less expensive**, advance ($3D + 1$) dynamical fields, i.e. $N_e, u_{\parallel e/i}, T_{e/i}$,
 - ▶ Solve the drift-reduced Braginskii Eqs. ⇒ **Assume high collisionality and drift-reduced ordering**



Chang C. S. *et al.*, Nucl. Fusion 57, 116023 (2017).



M. Giacomin, EFTC Oral Contribution

Develop a Gyrokinetic model for the plasma periphery retaining:

- Arbitrarily large deviations from equilibrium $\delta f_a \sim f_{aM} \Rightarrow \text{Full-F}$
- EM fluctuations at all scales $\Rightarrow \text{Full FLR and polarization effects}$
- Valid at arbitrary collisionality $\Rightarrow \text{Full-F GK Coulomb collision operator}$
- Includes the relevant collisionless kinetic effects \Rightarrow e.g. Landau-damping, gyro-averaging magnetic trapping, particle drift resonance effects
- Efficient numerical approach $(3D + 1)$

The Gyrokinetic Model for the Plasma Periphery

- Introduce the fundamental small parameter

$$\epsilon = \frac{\rho}{L_P} \sim \frac{\rho}{L_\phi} \ll 1, \quad L_P^{-1} \sim \nabla_\perp \ln \langle P \rangle_t, \quad \epsilon_B \sim \rho \nabla_\perp \ln \hat{B} \ll \epsilon$$

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- Generalized electromagnetic GK orderings

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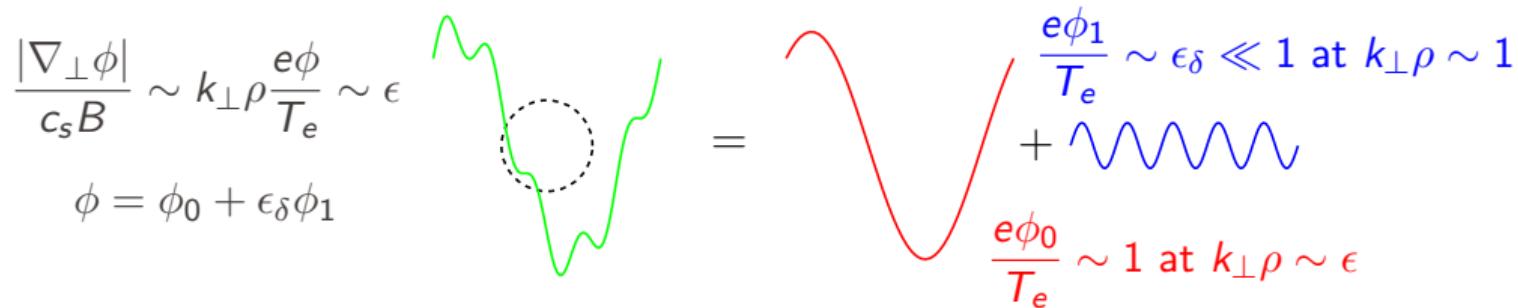
- Generalized electromagnetic GK orderings

$$\frac{|\nabla_\perp \phi|}{c_s B} \sim k_\perp \rho \frac{e\phi}{T_e} \sim \epsilon$$
$$\phi = \phi_0 + \epsilon_\delta \phi_1$$
$$\frac{e\phi_1}{T_e} \sim \epsilon_\delta \ll 1 \text{ at } k_\perp \rho \sim 1$$
$$\frac{e\phi_0}{T_e} \sim 1 \text{ at } k_\perp \rho \sim \epsilon$$

- Introduce the fundamental small parameter

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- Generalized electromagnetic GK orderings



- Remove fast compressional perturbations

$$\frac{|\delta \mathbf{B}_\perp|}{B} \sim \epsilon, \quad \delta \mathbf{B}_\perp \simeq \nabla A_\parallel \times \hat{\mathbf{b}}, \quad \hat{\mathbf{B}} = \nabla \times \hat{\mathbf{A}}, \quad A_\parallel = A_{\parallel 0} + \epsilon_\delta A_{\parallel 1},$$

- Collisional ordering, $\epsilon_\nu = \nu_{ii}/\Omega_i \sim \epsilon^2$

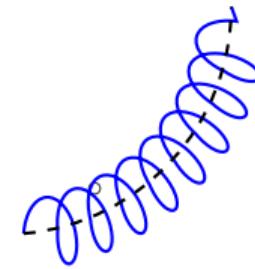
- **Dynamical reduction** from particle phase-space $\mathbf{z}^\nu = (\mathbf{x}, \mathbf{v})$ to gyrocenter phase-space $\overline{\mathbf{Z}}^\nu = (\overline{\mathbf{R}}, \overline{\mu}, \overline{v}_\parallel, \overline{\theta})$, with $\dot{\overline{\mu}} = 0$ up to $O(\epsilon^2, \epsilon_\delta^2)$

$$\gamma_a(\mathbf{x}, \mathbf{v}) = (q_a \mathbf{A} + m_a \mathbf{v}) \cdot d\mathbf{x} - \left[q_a \phi + \frac{m_a v^2}{2} \right] dt$$

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$$\langle \chi \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\bar{\theta} \chi \quad \downarrow$$



$$\bar{\Gamma}_a(\bar{\mathbf{R}}, \bar{v}_\parallel, \bar{\mu}, t) = q_a \mathbf{A}^* \cdot d\bar{\mathbf{R}} + \frac{B\bar{\mu}}{\Omega_a} d\bar{\theta} - [\mathcal{H}_0 + q_a \langle \Psi_1 \rangle] dt, \quad \partial_{\bar{\theta}} \bar{\Gamma}_a = 0$$

- Guiding-center Hamiltonian, \mathcal{H}_0
- Non-linear electromagnetic gyrokinetic potential

$$\langle \Psi_1 \rangle \sim \langle \Phi_1 \rangle + \left(\left\langle A_{\parallel 1}^2 \right\rangle - \partial_\mu \left[\left\langle \Phi_1^2 \right\rangle - \langle \Phi_1 \rangle^2 \right] \right), \quad \Phi_1 = \phi_1 - \bar{v}_\parallel A_{\parallel 1}$$

- From $\bar{\Gamma}_a$, single gyrocenter dynamics in the periphery

Free \parallel streaming

Cross-field drifts

Gyrokinetic transport

$$\dot{\bar{R}} = \bar{v}_{\parallel} \mathbf{b} + \mathbf{u}_E + \mathbf{U}_{\nabla} + \mathbf{U}_{\kappa} + \mathbf{U}_{pol} + \frac{\mathbf{b}}{B_{\parallel}^*} \times \bar{\nabla} \langle \Psi_1 \rangle_{\bar{R}} + \mathbf{U}_{NL}$$

$$m_a \dot{\bar{v}}_{\parallel} = q E_{\parallel} - q \frac{\mathbf{B}^*}{B_{\parallel}^*} \cdot \bar{\nabla} \langle \Psi_1 \rangle_{\bar{R}} - \bar{\mu} \mathbf{b} \cdot \bar{\nabla} B + F_{\parallel, NL}$$

$$\dot{\bar{\mu}} = 0$$

Magnetic trapping bounce motion

- Phase-space preserving characteristics, i.e. $d_t B_{\parallel}^* = 0$

- The $3D2V + 1$ gyrokinetic Boltzmann equation, $\overline{F_a} = \langle \overline{F_a} \rangle + \widetilde{\overline{F_a}}$

$$\frac{\partial}{\partial t} \left(B_{\parallel}^* \langle \overline{F_a} \rangle \right) + \bar{\nabla} \cdot \left(B_{\parallel}^* \dot{\overline{R}} \langle \overline{F_a} \rangle \right) + \frac{\partial}{\partial \bar{v}_{\parallel}} \left(B_{\parallel}^* \dot{\bar{v}}_{\parallel} \langle \overline{F_a} \rangle \right) = B_{\parallel}^* \sum_b C_{ab}$$

- Full Coulomb Collision operator

$$C_{ab} = \nu_{ab} \nabla_{\mathbf{v}} \cdot \left[\nabla_{\mathbf{v}} \nabla_{\mathbf{v}} G_b(\mathbf{v}) f_a(\mathbf{v}) - \frac{m_a}{m_b} f_a(\mathbf{v}) \nabla_{\mathbf{v}} H_b(\mathbf{v}) \right]$$

- **Self-Consistent** GK field equations by **Variational Approach**
- A short reminder of **Variational Approach**:
 - ▶ In particle phase-space $\mathbf{z}^\nu = (\mathbf{x}, \mathbf{v})$

$$\mathcal{A}[\phi] = \mathcal{A}_f + \mathcal{A}_p = \int dt \int \frac{d\mathbf{x}}{8\pi} |-\nabla\phi|^2 + \sum_a \int dt \int \mathbf{v} \gamma_a[\phi] f_a(\mathbf{x}, \mathbf{v})$$

$$\frac{\delta \mathcal{A}}{\delta \phi} \circ \check{\phi} = 0 \quad \Rightarrow -\nabla^2 \phi = 4\pi \sum_a \int d\mathbf{v} f_a(\mathbf{x}, \mathbf{v}) \frac{\delta \gamma_a}{\delta \phi},$$

- **Gyrokinetic Variational Principle** for the dynamical fields $(\phi_0, \phi_1, \hat{\mathbf{A}}, A_{\parallel 0}, A_{\parallel 1})$

$$\begin{aligned} \mathcal{A}[\phi_0, \phi_1, \hat{\mathbf{A}}, A_{\parallel 0}, A_{\parallel 1}] &= \mathcal{A}_f + \mathcal{A}_p = \int dt \int \frac{d\mathbf{x}}{8\pi} \left(|\mathbf{E}|^2 - |\hat{\mathbf{B}} + \delta\mathbf{B}_{\perp}|^2 \right) \\ &\quad + \int dt \sum_a \int d\mathbf{Z} \langle \bar{F}_a \rangle \bar{\Gamma}_a[\phi_0, \phi_1, \hat{\mathbf{A}}, A_{\parallel 0}, A_{\parallel 1}] \end{aligned}$$

$$\mathbf{E} = -\bar{\nabla}(\phi_0 + \phi_1) - \partial_t(\hat{\mathbf{A}} + \hat{\mathbf{b}}(A_{\parallel 0} + A_{\parallel 1})), \quad \delta\mathbf{B}_{\perp} \simeq \bar{\nabla}(A_{\parallel 0} + A_{\parallel 1}) \times \hat{\mathbf{b}}$$

- Field dynamics from **Least action principle**

$$\delta\mathcal{A} = \frac{\delta\mathcal{A}}{\delta\phi_0} \circ \check{\phi}_0 + \frac{\delta\mathcal{A}}{\delta\phi_1} \circ \check{\phi}_1 + \frac{\delta\mathcal{A}}{\delta\hat{\mathbf{A}}} \circ \check{\hat{\mathbf{A}}} + \frac{\delta\mathcal{A}}{\delta A_{\parallel 0}} \circ \check{A}_{\parallel 0} + \frac{\delta\mathcal{A}}{\delta A_{\parallel 1}} \circ \check{A}_{\parallel 1} = 0$$

- Coupled set of field equations \Rightarrow **GKPI/GKPII**, and **GKAI/GKAII/GKAIII**.

- **Gyrokinetic Boltzmann Equation**

$$\frac{\partial}{\partial t} \left(B_{\parallel}^* \langle \overline{F_a} \rangle \right) + \bar{\nabla} \cdot \left(B_{\parallel}^* \dot{\bar{\mathbf{R}}} \langle \overline{F_a} \rangle \right) + \frac{\partial}{\partial \bar{v}_{\parallel}} \left(B_{\parallel}^* \bar{v}_{\parallel} \langle \overline{F_a} \rangle \right) = B_{\parallel}^* \sum_b C_{ab} \mathbf{r}_a$$

- **Self-consistent GK field Equations**

$$\bar{\nabla} \cdot \mathbf{E} = 4\pi \sum_a \int d\mathbf{v} \langle \overline{F_a} \rangle \frac{\delta \bar{\Gamma}_a}{\delta \phi_0}$$

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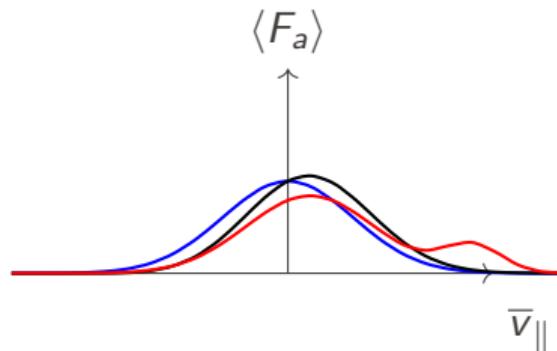
$$\bar{\nabla} \times (\hat{\mathbf{B}} + \delta \mathbf{B}_{\perp}) = 4\pi \sum_a \int d\mathbf{v} \langle \overline{F_a} \rangle \dots$$

⋮

- Still **3D2V + 1** and **Full-F** ...

- From $3D2V + 1$ to $3D + 1$ and **Full-F** \Rightarrow **Hermite-Laguerre moment expansion**

$$\langle \bar{F}_a \rangle = F_{aM} \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} N_a^{pj}(\mathbf{R}, t) H_p(\bar{v}_{\parallel}) L_j(v_{\perp}^2), \quad N_a^{pj} \Rightarrow \text{Gyro-moments}$$



(0,0): F_{aM}

(1,0): $F_{aM} \times (1 + N_a^{10} \bar{v}_{\parallel})$

(P,0): $F_{aM} \times \sum_{p=0}^{P_a} N_a^{p0} H_p(\bar{v}_{\parallel})$

Multi-Fidelity

- Apply **Hermite-Laguerre projector** $\|\cdot\|_a^{pj}$ to gyrokinetic Boltzmann equation

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- **$3D + 1$ Moment Hierarchy equations** for $N_a^{pj}(R, t)$

$$\frac{\partial}{\partial t} N_a^{pj} + \bar{\nabla} \cdot \left\| \dot{R} \right\|_a^{pj} - \frac{\sqrt{2l}}{v_{th\parallel a}} \left\| \dot{v}_{\parallel} \right\|_a^{p-1j} + \mathcal{F}_a^{pj} = \sum_b C_{ab}^{pj}$$

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Temporal Evolution

- Apply **Hermite-Laguerre projector** $\|\cdot\|_a^{pj}$ to gyrokinetic Boltzmann equation
- **3D + 1 Moment Hierarchy equations** for $N_a^{pj}(R, t)$

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↑
Temporal Evolution
↑
Spatial Evolution

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Temporal Evolution

Spatial Evolution

Parallel Forces for $p \geq 1$

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- **3D + 1 Moment Hierarchy equations** for $N_a^{pj}(R, t)$

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Temporal Evolution

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Parallel Forces for $p \geq 1$

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Fluid Operator
 $(T_a, \bar{u}_\parallel)_a$

Spatial Evolution

- Apply **Hermite-Laguerre projector** $\|\cdot\|_a^{pj}$ to gyrokinetic Boltzmann equation
- **$3D + 1$ Moment Hierarchy equations** for $N_a^{pj}(R, t)$

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Temporal Evolution Parallel Forces for $p \geq 1$ Fluid Operator $(T_a, \bar{u}_\parallel a)$

- Hermite-Laguerre representation of gyrocenter dynamics

$$\left\| \dot{\bar{R}} \right\|_a^{pj} = \sum_{l,k} \left[\mathbf{U}_0 \delta_l^p \delta_k^j + \bar{v}_{th\parallel a} \mathcal{V}_{lk}^{pj} \mathbf{b} + \mathcal{M}_{lk}^{pj} \mathbf{U}_\nabla + \dots \right] N_a^{lk} + \left\| \frac{\mathbf{b}}{B_\parallel^*} \times \bar{\nabla} \langle \Psi_1 \rangle \right\|_a^{pj}$$

$$m_a \left\| \dot{\bar{v}}_\parallel \right\|_a^{p-1j} = \sum_{l,k} \left[F_{\parallel a} \delta_l^{p-1} \delta_k^j - F_{Ma} \mathcal{M}_{lk}^{p-1j} + \dots \right] N_a^{lk} - q_a \left\| \frac{\mathbf{B}^*}{B_\parallel^*} \cdot \bar{\nabla} \langle \Psi_1 \rangle \right\|_a^{p-1j}$$

■ Hermite-Laguerre representation of gyrocenter dynamics

$$\left\| \dot{\bar{R}} \right\|_a^{pj} = \sum_{l,k} \left[\mathbf{U}_0 \delta_l^p \delta_k^j + \bar{v}_{th\parallel a} \mathcal{V}_{lk}^{pj} \mathbf{b} + \mathcal{M}_{lk}^{pj} \mathbf{U}_\nabla + \dots \right] N_a^{lk} + \left\| \frac{\mathbf{b}}{B_\parallel^*} \times \bar{\nabla} \langle \Psi_1 \rangle \right\|_a^{pj}$$

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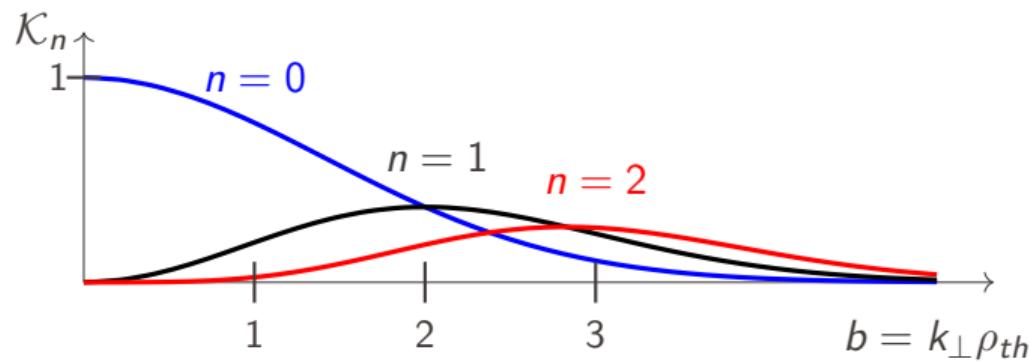
► \parallel (parallel streaming) and \perp (gyro-averaging, magnetic gradient) phase-mixing

$$\sum_{l,k} \mathcal{V}_{lk}^{pj} N_a^{lk} = \sqrt{\frac{p+1}{2}} N_a^{p+1j} + \sqrt{\frac{p}{2}} N_a^{p-1k}$$

$$\sum_{l,k} \mathcal{M}_{lk}^{pj} N_a^{lk} = (2j+1) N_a^{pj+1} - j N_a^{pj-1} - (j-1) N_a^{pj}$$

■ Fourier-Laguerre expansion of FLR terms

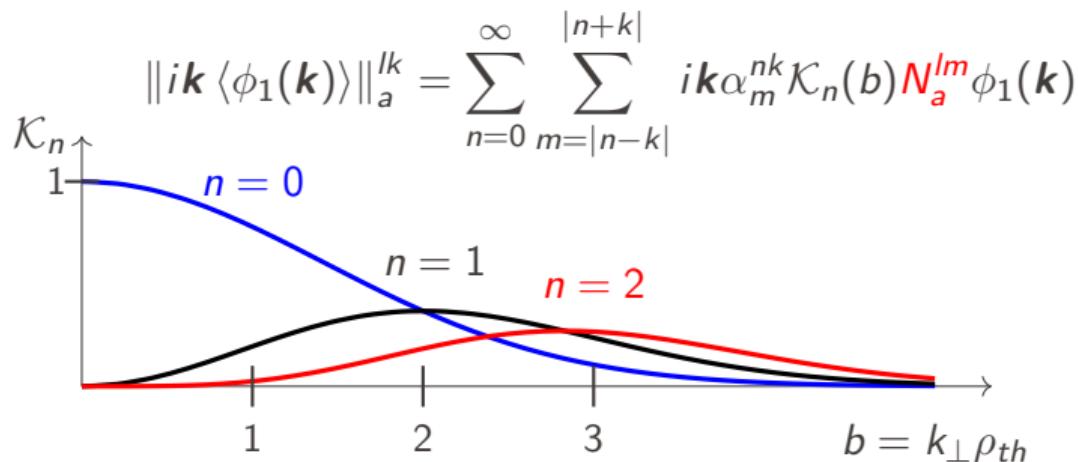
$$\langle \Psi_1(\mathbf{k}) \rangle = J_0 \left(\frac{k_\perp v_\perp}{\Omega_a} \right) \Psi_1(\mathbf{k}) = \sum_{n=0}^{\infty} \mathcal{K}_n(b) L_n(\bar{v}_\perp^2) \Psi_1(\mathbf{k}), \quad \mathcal{K}_n(b) = \frac{1}{n!} \left(\frac{b}{2} \right)^{2n} e^{-b^2/2}$$



- Fourier-Laguerre expansion of FLR terms

$$\langle \Psi_1(\mathbf{k}) \rangle = J_0 \left(\frac{k_\perp v_\perp}{\Omega_a} \right) \Psi_1(\mathbf{k}) = \sum_{n=0}^{\infty} \mathcal{K}_n(b) L_n(\bar{v}_\perp^2) \Psi_1(\mathbf{k}), \quad \mathcal{K}_n(b) = \frac{1}{n!} \left(\frac{b}{2} \right)^{2n} e^{-b^2/2}$$

- In electrostatic and homogenous plasma (generalized to NL EM and inhomogenous case)



■ Spatial and time evolution of gyrocenter fluid moments

$$(p,j) = (0,0) : \frac{1}{N_a} \frac{d_a}{dt} N_a = \sum_b C_{ab}^{00} - \frac{\partial}{\partial t} \left(\frac{\bar{B}_{\parallel a}^*}{B} \right) - \bar{\nabla} \cdot \mathbf{u}_a^0$$

$$(p,j) = (1,0) : \frac{d_a}{dt} \bar{u}_{\parallel a} = \frac{\bar{v}_{th\parallel a}}{\sqrt{2}} \sum_b C_{ab}^{10} - \frac{1}{N_a \sqrt{2}} \bar{\nabla} \cdot \left(\mathbf{u}_a^{\parallel 1} N_a \bar{v}_{th\parallel a} \right) - \|\dot{\bar{v}_{\parallel}}\|_a^{00} + \dots$$

$$(p,j) = (0,1) : \frac{d_a}{dt} \left(\frac{\bar{T}_{\perp a}}{B} \right) = \frac{N_a \bar{T}_{\perp a}}{B} \sum_b C_{ab}^{01} - \bar{\nabla} \cdot \left(\frac{\mathbf{u}_a^{\perp 1} \bar{P}_{\perp a}}{B} \right) + \dots$$

$$(p,j) = (2,0) : \frac{N_a}{\sqrt{2}} \frac{d_a}{dt} \bar{T}_{\parallel a} = \bar{P}_{\parallel a} \sum_b C_{ab}^{20} - \bar{\nabla} \cdot \left(\bar{T}_{\parallel a} N_a \mathbf{u}_a^{\parallel 2} \right) + \dots$$

$$\vdots$$

■ Spatial and time evolution of gyrocenter fluid moments

$$(p,j) = (0,0) : \frac{1}{N_a} \frac{d_a}{dt} N_a = \sum_b C_{ab}^{00} - \frac{\partial}{\partial t} \left(\frac{\bar{B}_{\parallel a}^*}{B} \right) - \bar{\nabla} \cdot \mathbf{u}_a^0$$

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- High-collisionality \Rightarrow **Improved Braginskii equations** (Jorge *et al.* (2017))
- **Semi-collisional closure** at arbitrary collisionality (Loureiro *et al.* (2011))

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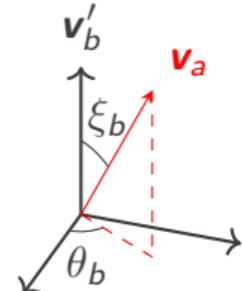
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■ Spherical harmonic multipole expansion of full Coulomb¹

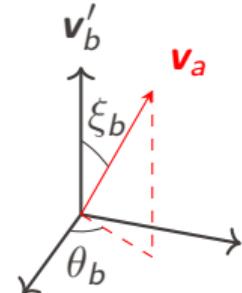
$$H_b(\mathbf{v}) = \int d\mathbf{v}' \frac{f_b(\mathbf{v}')}{|\mathbf{v}' - \mathbf{v}|}, \quad G_b(\mathbf{v}) = \int d\mathbf{v}' f_b(\mathbf{v}') |\mathbf{v}' - \mathbf{v}|$$



¹R. Jorge, B. J. Frei, and P. Ricci, "Non-Linear Gyrokinetic Coulomb Collision Operator", To appear in JPP, arXiv:1906.03252 [physics.plasm-ph] (2019).

- Spherical harmonic multipole expansion of full Coulomb ¹

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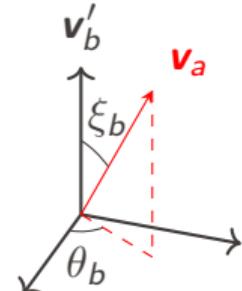
- Exact non-linear evaluation of gyro-average \Rightarrow Full FLR effects

$$\langle C_{ab} \rangle = \nu_{ab} \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \sum_{r,s,p,q,\dots} \dots F_{mab}^{mrspq}(\bar{\mu}, \bar{v}_{\parallel}) \langle Y_l^m(\xi, \theta) N_a^{rs}(x) N_b^{pq}(x) \rangle$$

¹R. Jorge, B. J. Frei, and P. Ricci, "Non-Linear Gyrokinetic Coulomb Collision Operator", To appear in JPP, arXiv:1906.03252 [physics.plasm-ph] (2019).

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- Fourier-Hermite-Laguerre gyro-moment expansion

$$C_{ab}^{pj} = \nu_{ab} \sum_{\mathbf{k}, \mathbf{k}'} \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \sum_{r,s,p,q,\dots} \mathcal{K}_s(b_a) \mathcal{K}_q(b_a) N_a^{rs}(\mathbf{k}) N_b^{pq}(\mathbf{k}')$$

- Accurate at arbitrary collisionality and Multi-species

¹R. Jorge, B. J. Frei, and P. Ricci, "Non-Linear Gyrokinetic Coulomb Collision Operator", To appear in JPP, arXiv:1906.03252 [physics.plasm-ph] (2019).

- **GKPI** from $\delta\mathcal{A}/\delta\phi_0 \circ \check{\phi}_0 = 0$, and **GKPII** from $\delta\mathcal{A}/\delta\phi_1 \circ \check{\phi}_1 = 0$

$$\bar{\nabla} \cdot \mathbf{E} = 4\pi \sum_a [q_a N_a - \bar{\nabla} \cdot \mathcal{P}_a^I], \quad \bar{\nabla} \cdot \mathbf{E} = 4\pi \sum_a [q_a \langle N_a \rangle + \mathcal{P}_a^{II}]$$

- **Self-consistent Full-F polarization effects** from dynamical reduction

$$\mathcal{P}_a^I \simeq \frac{m_a N_a}{B} \mathbf{b} \times \left[\left\| \dot{\mathbf{R}} \right\|_a^{00} - \mathbf{u}_E N_a \right] - \frac{\mathbf{b}}{B} \times \bar{\nabla} \times \left[\frac{\bar{P}_{\perp a}}{2\Omega_a} \mathbf{b} \right]$$

$$\mathcal{P}_a^{II} = \sum_{n,r,s,\dots} \mathcal{K}_n(b) \mathcal{K}_r(b'^\dagger) \left[\phi_1(\mathbf{k}) \left(\frac{N_a \textcolor{red}{N_a^{0s}}}{\bar{T}_{\perp a}} \right) (\mathbf{k}') - A_{\parallel 1}(\mathbf{k}) \left(\frac{\textcolor{red}{J_{\parallel a}^{1s}}}{\bar{T}_{\perp a}} \right) (\mathbf{k}') \right] \dots$$

- **Two coupled field equations** for ϕ_0, ϕ_1
- **Recover LWL polarization density**

$$\mathcal{P}_a^{II} \simeq \frac{q_a^2}{m_a} \bar{\nabla} \cdot \left(\frac{N_a}{\Omega_a^2} \bar{\nabla}_\perp \phi_1 \right) - \frac{q_a}{m_a} \bar{\nabla} \cdot \left(\frac{\textcolor{red}{J_{\parallel a}^{10}}}{\Omega_a^2} \bar{\nabla}_\perp A_{\parallel 1} \right)$$

- **GKAI** from $\delta\mathcal{A}/\delta A_{\parallel 0} \circ \check{A}_{\parallel 0} = 0$, and **GKAI** from $\delta\mathcal{A}/\delta A_{\parallel 1} \circ \check{A}_{\parallel 1} = 0$

$$\begin{aligned}\left[\bar{\nabla} \times (\hat{\mathbf{B}} + \delta \mathbf{B}_\perp) - \partial_t \mathbf{E}\right] \cdot \hat{\mathbf{b}} &= 4\pi \sum_a \left[\mathbf{J}_a + \partial_t \mathcal{P}_a^I + \bar{\nabla} \times (\mathcal{M}_{a1}^I + \mathcal{M}_{a2}^I) \right] \cdot \hat{\mathbf{b}} \\ \left[\bar{\nabla} \times (\hat{\mathbf{B}} + \delta \mathbf{B}_\perp) - \partial_t \mathbf{E}\right] \cdot \hat{\mathbf{b}} &= 4\pi \sum_a [\langle J_{\parallel a} \rangle + \mathcal{M}_a^{II}]\end{aligned}$$

- **Self-consistent Full-F magnetizations effects** from dynamical reduction

$$\begin{aligned}\mathcal{M}_{a1}^I &\simeq -\frac{\bar{P}_{\perp a} B}{B^2} (N_a^{00} - N_a^{01}), \quad \mathcal{M}_{a2}^I \simeq \frac{m_a N_a v_{th\parallel a}}{\sqrt{2} B} \left(\left\| \dot{\bar{\mathbf{R}}} \right\|_a^{10} - N_a^{10} u_E \right) \\ \mathcal{M}_a^{II} &\simeq q_a \sum_{n,r,s,\dots} \mathcal{K}_n(b) \mathcal{K}_r(b'^\dagger) \left[\phi_1(\mathbf{k}) \left(\frac{J_{\parallel a}^{1s}}{\bar{T}_{\perp a}} \right) (\mathbf{k}') - A_{\parallel 1}(\mathbf{k}) \left(\frac{J_{\parallel a}^{2s}}{\bar{T}_{\perp a}} \right) (\mathbf{k}') \right] \dots\end{aligned}$$

- **Two coupled field equations** for $A_{\parallel 0}, A_{\parallel 1}$
- **Recover LWL magnetization current**

- From $\hat{\mathbf{B}} \times (\delta\mathcal{A}/\delta\hat{\mathbf{A}} \circ \check{\hat{\mathbf{A}}}) = 0$ with $\mathcal{M}'_{a1} \simeq -\frac{\bar{P}_{\perp a}\hat{\mathbf{B}}}{\hat{B}^2}$

$$\sum_a N_a m_a \frac{d^0}{dt} \mathbf{U}_0 \Big|_{\perp} = \frac{1}{4\pi} \left[\left(\hat{\mathbf{B}} \cdot \bar{\nabla} \right) \hat{\mathbf{B}} - \frac{\bar{\nabla} \hat{B}^2}{2} - \hat{\mathbf{B}} \times (\bar{\nabla} \times \delta \mathbf{B}_{\perp}) \right] - \bar{\nabla} \cdot \Pi_{CGL}$$

- Steep pressure gradient** with large scale $\hat{\mathbf{B}}$ consistent with **low- β plasma**, i.e.

$$\epsilon_B = \frac{\rho}{L_B} \sim \beta \frac{\rho}{L_P} \ll \epsilon$$

GK Moment Hierarchy

$$\partial_t N_a^{pj} + \bar{\nabla} \cdot \left\| \dot{\bar{R}} \right\|_a^{pj} - \frac{\sqrt{2I}}{v_{th||a}} \left\| \dot{\bar{v}}_{||} \right\|_a^{p-1j} + \mathcal{F}_a^{pj} = \sum_b C_{ab}^{pj}$$

GKPI

$$\bar{\nabla} \cdot \mathbf{E} = 4\pi \sum_a [q_a N_a - \bar{\nabla} \cdot \mathcal{P}_a^I]$$

GKPII

$$\bar{\nabla} \cdot \mathbf{E} = 4\pi \sum_a [q_a \langle N_a \rangle + \mathcal{P}_a^{II}]$$

GKAI

$$[\bar{\nabla} \times (\hat{\mathbf{B}} + \delta \mathbf{B}_{\perp}) - \partial_t \mathbf{E}] \cdot \hat{\mathbf{b}} = 4\pi \sum_a [\mathbf{J}_a + \partial_t \mathcal{P}_a^I + \bar{\nabla} \times (\mathcal{M}_{a1}^I + \mathcal{M}_{a2}^I)] \cdot \hat{\mathbf{b}}$$

GKAII

$$[\bar{\nabla} \times (\hat{\mathbf{B}} + \delta \mathbf{B}_{\perp}) - \partial_t \mathbf{E}] \cdot \hat{\mathbf{b}} = 4\pi \sum_a [\langle J_{||a} \rangle + \mathcal{M}_a^{II}]$$

GKAIII/Pressure Balance Equation

$$\sum_a N_a m_a d_t^0 U_0 \Big|_{\perp} = \frac{1}{4\pi} \left[(\hat{\mathbf{B}} \cdot \bar{\nabla}) \hat{\mathbf{B}} - \frac{\bar{\nabla} \hat{B}^2}{2} - \hat{\mathbf{B}} \times (\bar{\nabla} \times \delta \mathbf{B}_{\perp}) \right] - \bar{\nabla} \cdot \boldsymbol{\Pi}_{CGL}$$

Gyrokinetic Ion-Temperture Gradient η_i -Drift Instability

- Slab electrostatic Ion Temperature gradient
- Linearized GK Moment Hierarchy equation + GKPII

$$\begin{aligned}
 & \frac{\partial}{\partial t} N_i^{pj} + ik_{\parallel} \sqrt{\tau_i} \left(\sqrt{p+1} N_i^{p+1j} + \sqrt{p} N_i^{p-1j} \right) \\
 & - i \mathcal{K}_j(b_i) \omega_* \phi_1 \delta_{p,0} - i \eta_j \omega_* \phi \left[\mathcal{K}_j(b_i) \left(\delta_{p,2}/\sqrt{2} - \delta_{p,0} \right) \right. \\
 & \left. + \delta_{p,0} ((2j+1)\mathcal{K}_j(b_i) - j\mathcal{K}_{j-1}(b_i) - (j+1)\mathcal{K}_{j+1}(b_i)) \right] \\
 & + \frac{1}{\sqrt{\tau_i}} ik_{\parallel} \mathcal{K}_j(b_i) \phi_1 \delta_p^1 = -\nu(p+2j+a_i) N_i^{pj} \\
 & \frac{1}{\tau_i} \left[1 - \sum_{n=0} \mathcal{K}_n^2(b_i) N_i^{0n} \right] \phi_1 = \sum_n \mathcal{K}_n(b_i) N_i^{0n}
 \end{aligned}$$

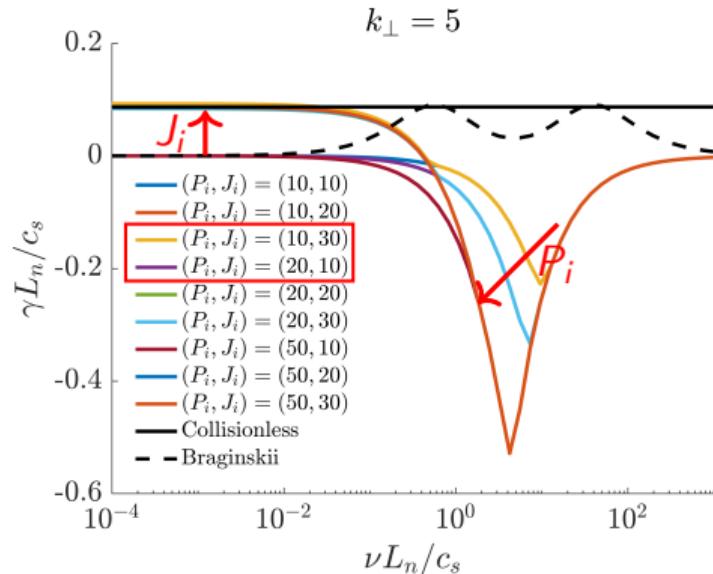
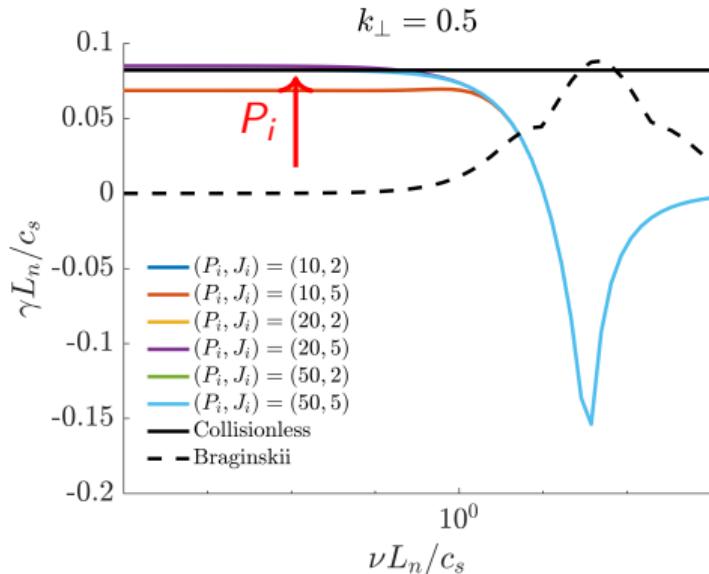
- Truncation closure at $(p,j) = (P_i, J_i)$: $\gamma N_a^{pj} = (\mathbf{M}_a + i\mathbf{C}_a) N_a^{pj}$

- Slab electrostatic Ion Temperature gradient
- Linearized GK Moment Hierarchy equation + GKPII

$$\begin{aligned}
 \frac{\partial}{\partial t} N_i^{pj} + ik_{\parallel} \sqrt{\tau_i} \left(\sqrt{p+1} N_i^{p+1j} + \sqrt{p} N_i^{p-1j} \right) &\quad \xleftarrow{i\bar{v}_{\parallel} k_{\parallel} \delta f_i} \\
 - i\mathcal{K}_j(b_i) \omega_* \phi_1 \delta_{p,0} - i\eta_j \omega_* \phi \left[\mathcal{K}_j(b_i) \left(\delta_{p,2}/\sqrt{2} - \delta_{p,0} \right) \right. &\quad \xrightarrow{\frac{i\mathbf{b} \times \mathbf{k}_{\perp} J_0 \phi_1}{B} \cdot \bar{\nabla}_{\perp} F_{iM}} \\
 \left. + \delta_{p,0} ((2j+1)\mathcal{K}_j(b_i) - j\mathcal{K}_{j-1}(b_i) - (j+1)\mathcal{K}_{j+1}(b_i)) \right] & \\
 ik_{\parallel} J_0 \phi_1 F_{iM} & \quad \curvearrowleft \\
 + \frac{1}{\sqrt{\tau_i}} ik_{\parallel} \mathcal{K}_j(b_i) \phi_1 \delta_p^1 &= -\nu(p+2j+a_i) N_i^{pj} & \quad \curvearrowleft \\
 \frac{1}{\tau_i} \left[1 - \sum_{n=0} \mathcal{K}_n^2(b_i) N_i^{0n} \right] \phi_1 &= \sum_n \mathcal{K}_n(b_i) N_i^{0n} & \quad \curvearrowleft C_{ii}^{pj}
 \end{aligned}$$

- Truncation closure at $(p,j) = (P_i, J_i)$: $\gamma N_a^{pj} = (\mathbf{M}_a + i\mathbf{C}_a) N_a^{pj}$

- Retrieves sITG collisionless growth rate



- FLR requires $J_i \gtrsim k_\perp^2$
- GK Coulomb coll. to Braginskii limit (at low k_\perp , i.e. drift-reduced)

Conclusion

To summarize:

- A Gyrokinetic moment-based model
 - ▶ Full-F
 - ▶ Retains the essential **kinetic effects** (e.g. Landau damping, gyro-averaging, particle drift resonance)
 - ▶ **Accurate** collisional effects (Multi-species, FLR effects)
 - ▶ **Self-consistent** Gyrokinetic field equations
- sITG Model application; extended to include finite β and toroidal effects, and arbitrary collisionality
- Ongoing first numerical investigations
- Results to benchmark against existing GK codes

Backup slides

- Ion EM Linearized Moment Hierarchy,

$$\begin{aligned}
 & \frac{\partial}{\partial t} N_i^{pj} + ik_{\parallel} \sqrt{\tau_i} \left(\sqrt{p+1} N_i^{p+1j} + \sqrt{p} N_i^{p-1j} \right) - \sum_n \mathcal{K}_n(b_i) i\omega_* \left(\delta_{j,n} \delta_{p,0} + \eta_i \mathcal{T}_{1n}^{pj} \right) \phi \\
 & + \sqrt{2\tau_i} \sum_n \mathcal{K}_n(b_i) i\omega_* \left(\frac{\delta_{j,n} \delta_{p,1}}{\sqrt{2}} + \eta_i \mathcal{T}_{2n}^{pj} \right) \psi \\
 & + \frac{1}{\sqrt{\tau_i}} ik_{\parallel} \mathcal{K}_j(b_i) \left(\phi \delta_{p,1} - \sqrt{2\tau_i} \psi \left(\frac{\delta_{p,0}}{\sqrt{2}} + \delta_{p,2} \right) \right) = \sum_s C_{is}^{pj}, \quad \delta \mathbf{B}_{\perp} \simeq \psi \mathbf{k}_{\perp} \times \mathbf{b}
 \end{aligned}$$

- Linearized GK Poisson and Ampere's law,

$$\left[\alpha_D + \frac{1}{\tau_i} \left(1 - \sum_n \mathcal{K}_n^2(b_i) \right) \right] \phi = \sum_n \mathcal{K}_n(b_i) N_i^{0n},$$

$$\left[\frac{k_{\perp}^2}{\beta_e} + \sum_n \mathcal{K}_n^2(b_i) \right] \psi = \sqrt{\tau_i} \sum_n \mathcal{K}_n(b_i) N_i^{1n}.$$

- Magnetic trapping, toroidal magnetic drift resonance effects

$$\begin{aligned}
 & \frac{\partial}{\partial t} N_a^{pj} + \frac{ik_{\parallel}\sqrt{\tau_a}}{\sigma_a} \left(\sqrt{p+1}N_a^{p+1j} + \sqrt{p}N_a^{p-1j} \right) \\
 & + \frac{\tau_a}{q_a} (i\omega_{\kappa a}) \left(\sqrt{(p+1)(p+2)}N_a^{p+2j} + (2p+1)N_a^{pj} + \sqrt{p(p-1)}N_a^{p-2j} \right) \\
 & + \frac{\tau_a}{q_a} (i\omega_{\nabla a}) \left((2j+1)N_a^{pj} - jN_a^{pj-1} - (j+1)N_a^{pj+1} \right) \\
 & + \frac{\sqrt{\tau_a}}{\sigma_a} \bar{\nabla}_{\parallel} \ln B \sqrt{p} \left((2j+1)N_a^{p-1j} - jN_a^{p-1j-1} - (j+1)N_a^{p-1j+1} \right) \\
 & - \sum_n \mathcal{K}_n(b_a) ik_{\perp} \eta_n \left(\delta_{n,j} \delta_{p,0} + \eta_{T_a} \mathcal{T}_{1n}^{pj} \right) \phi + \frac{\sqrt{2\tau_a}}{\sigma_a} \sum_n \mathcal{K}_n(b_a) ik_{\perp} \eta_n \left(\frac{\delta_{j,n} \delta_{p,1}}{\sqrt{2}} + \eta_{T_a} \mathcal{T}_{2n}^{pj} \right) \psi \\
 & + \frac{q_a}{\sigma_a \sqrt{\tau_a}} ik_{\parallel} \mathcal{K}_j(b_a) \left(\phi \delta_p^1 - \frac{\sqrt{2\tau_a}}{\sigma_a} \psi \left(\frac{\delta_p^0}{\sqrt{2}} + \delta_p^2 \right) \right) = C_a^{pj}.
 \end{aligned}$$

■ 1D periodic non-linear coupled fluid equations for N_e^I ,

$$I = 0 : \frac{1}{N_e} \frac{\partial N_e}{\partial t} = -\sqrt{\frac{T_{\parallel e}}{2}} u_{\parallel e} \nabla_{\parallel} N_e - \sqrt{2 T_{\parallel e}} \nabla_{\parallel} u_{\parallel e},$$

$$I = 1 : \frac{\partial u_{\parallel e}}{\partial t} = \frac{C_e^1}{\sqrt{2}} - \frac{\sqrt{T_{\parallel e}}}{2\sqrt{2}} \nabla_{\parallel} \ln N_e + \frac{1}{\sqrt{2 T_{\parallel e}}} \nabla_{\parallel} \phi + \dots,$$

$$I = 2 : \frac{1}{T_{\parallel e}} \frac{\partial T_{\parallel e}}{\partial t} = \frac{\sqrt{2} C_e^2}{N_e} - \sqrt{\frac{3}{2} T_{\parallel e}} N_e^3 \nabla_{\parallel} \ln N_e - \sqrt{6 T_{\parallel e}} \nabla_{\parallel} N_e^3 + \dots,$$

$$I > 2 : \frac{\partial N_e^I}{\partial t} = \frac{C_e^I}{N_e} + \sum_p \left[C_p^I N_e^p \nabla_{\parallel} \ln N_e + D_p^I N_e^p \nabla_{\parallel} \phi + I_p^I \nabla_{\parallel} N_e^p + \dots \right].$$

■ Mimic collisional dissipation,

► $C_e^0 = 0$, $C_e^1 = -\nu \sqrt{2} u_{\parallel e}$, ..., $C_e^I = -\nu \left(I N_e^I + \sqrt{2I} u_{\parallel e} N_e^{I-1} \right)$,

■ Truncation closure: $N_e^I = 0$ for $I > L$.

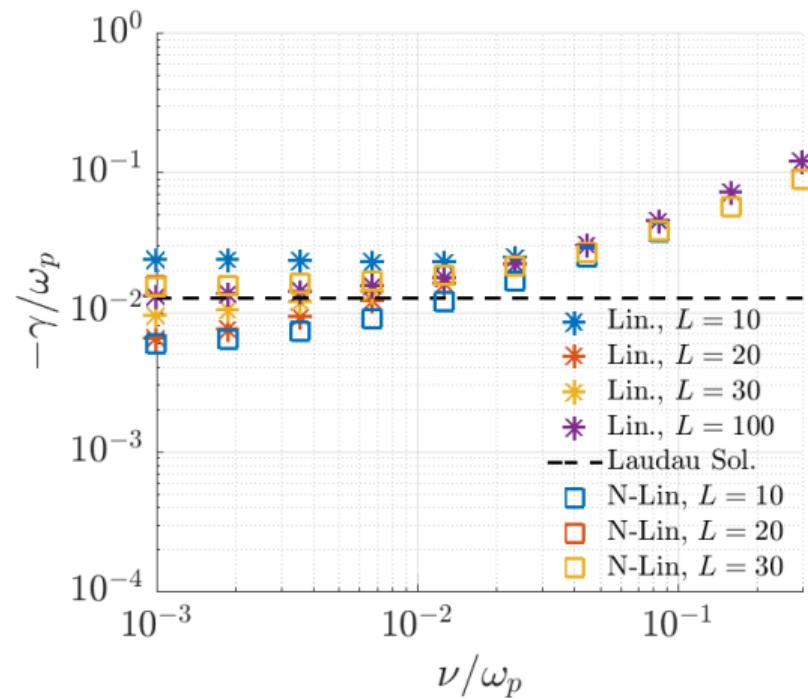
- Collisional damping, $\gamma \simeq \nu/2$

with $L = 2$,

- Lower N-Lin. L -resolution needed at $\nu/\omega_{pe} \ll 1$,

$$i\partial_t N_e^I - \left(\sqrt{I+1}N_e^{I+1} + \sqrt{I}N_e^{I-1} \right) / \sqrt{2}$$

$$- \frac{N_e^0 \delta_{I,1}}{\sqrt{2}(k_{\parallel} \lambda_D)^2} = -i\nu I N_e^I.$$



- Artificial "echos" due to truncation closure in $|N_e^l(t)|$,

